

PERIMETER UNDER MULTIPLE STEINER SYMMETRIZATIONS

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ABSTRACT. Steiner symmetrization along n linearly independent directions transforms every compact subset of \mathbb{R}^n into a set of finite perimeter.

Steiner symmetrization is a volume-preserving rearrangement that introduces a hyperplane of symmetry. A key property is that Steiner symmetrization strictly reduces the perimeter of every convex set that is not already reflection symmetric [5]. The perimeter of a non-convex set of finite perimeter decreases strictly under Steiner symmetrization in most directions, but not necessarily in all of them [3].

We seek to bound the perimeter of an arbitrary compact set $A \subset \mathbb{R}^n$ after a finite sequence of Steiner symmetrizations. Our main result is that n consecutive Steiner symmetrization in linearly independent directions suffice to transform A into a set of finite perimeter.

Theorem 1 (Perimeter estimate). *If $A \subset \mathbb{R}^n$ is a compact set and u_1, \dots, u_n are linearly independent unit vectors in \mathbb{R}^n , then*

$$(1) \quad \text{Per}(S_{u_n} \dots S_{u_1} A) \leq \frac{a_n R^{n-1}}{|\det(u_1, \dots, u_n)|},$$

where $a_n = 2n\omega_{n-1}$, and R is the outradius of A .

The theorem is motivated by the special case of the coordinate directions e_1, \dots, e_n . The set $S_{e_n} \dots S_{e_1} A$ is symmetric under reflection at each coordinate hyperplane, and its intersection with the positive cone lies under the graph of a monotone function $x_n = f(x_1, \dots, x_{n-1})$. The perimeter of such a set is bounded by twice the sum of the area of its projections onto the coordinate hyperplanes, which cannot exceed $a_n R^{n-1}$, see Figure 1.

We start with some definitions and notation. The dimension $n \geq 2$ will be fixed throughout the paper. The n -dimensional volume of a Lebesgue measurable set $A \subset \mathbb{R}^n$ is denoted by $\text{Vol}(A)$. By $\text{Per}(A)$,

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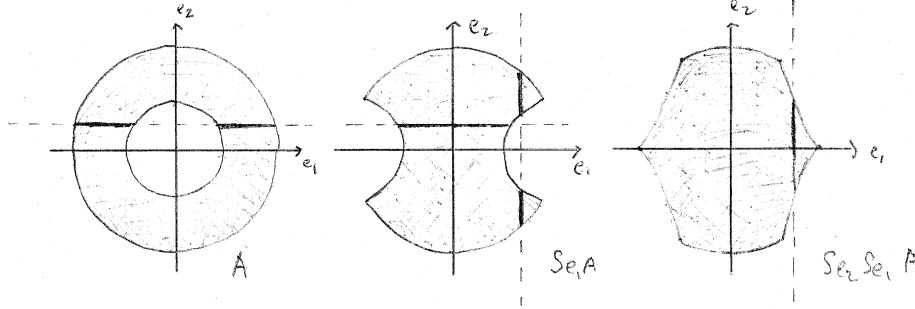


FIGURE 1. Steiner symmetrization along the coordinate directions

we mean the **Caccioppoli perimeter** of A , defined by

$$\text{Per}(A) = \sup_{\|F\|_\infty \leq 1} \int_A \text{div } F(x) \, dx,$$

where the supremum runs over all smooth compactly supported vector fields F on \mathbb{R}^n . If $\text{Per}(A) < \infty$, then its value agrees with the $(n-1)$ -dimensional Hausdorff measure of the essential boundary of A .

We denote by B the closed unit ball in \mathbb{R}^n , centered at the origin, and by ω_n its volume. The closed centered ball of radius $\rho > 0$ will be denoted by B_ρ . The **Minkowski sum** of two subsets $A, C \subset \mathbb{R}^n$ is given by

$$A + C = \{a + c : a \in A, c \in C\}.$$

Their **Minkowski difference** is the largest set whose Minkowski sum with C lies in A ,

$$A - C = \{x \in \mathbb{R}^n : x + c \in A \, \forall c \in C\}.$$

The sets $A + B_\rho$ and $A - B_\rho$ will be called the **outer** and **inner parallel sets** of A . The **Hausdorff distance** between A and C is given by

$$d_H(A, C) = \inf\{\delta > 0 : A \subset C + B_\delta \text{ and } C \subset A + B_\delta\}.$$

Let $A \subset \mathbb{R}^n$ be a compact set, and let $u \in \mathbb{R}^n$ be a unit vector. The **Steiner symmetrization** of A in the direction of u is defined by the following property. For each point $x \perp u$, we compute the length of the intersection of A with the inverse image of x under the orthogonal projection onto the hyperplane u^\perp , and then replace it with the closed interval of the same one-dimensional measure centered on u^\perp . If the intersection is empty, then the interval is empty; if it is a nonempty set of measure zero, then the interval consists of a single point. The resulting set will be denoted by $S_u A$. Note that $S_u A$ is compact and symmetric under reflection at u^\perp . By Cavalieri's principle, $S_u A$ has the

same volume as A , i.e., Steiner symmetrization is a volume-preserving rearrangement. The **symmetric rearrangement** of A is the closed centered ball A^* of the same volume as A . If A is empty, we take A^* to be empty; if A is a non-empty set of measure zero, then $A^* = \{0\}$. We will refer to the radius of A^* as the **volume radius** of A .

The corresponding symmetrizations of functions are defined by symmetrizing their level sets. Let f be a nonnegative continuous function with compact support. Its Steiner symmetrization $S_u f$ is determined by the property that

$$\{x : S_u f(x) \geq t\} = S_u \{x : f(x) \geq t\}$$

for every $t > 0$, and its symmetric decreasing rearrangement f^* is the unique radially decreasing continuous function that is equimeasurable to f . These symmetrizations improve the modulus of continuity and contract distances in the space of continuous functions.

It is useful to think of Steiner symmetrization as an operation on the one-dimensional cross sections

$$A(x) = \{t \in \mathbb{R} : x + tu \in A\}$$

for $x \in u^\perp$. By definition,

$$(S_u A)(x) = (A(x))^*,$$

where $(A(x))^*$ is the one-dimensional symmetric rearrangement of $A(x)$. Since one-dimensional symmetrization preserves the subset relation, Steiner symmetrization preserves it as well, and therefore

$$\begin{aligned} S_u A \cap S_u C &\supset S_u (A \cap C), \\ S_u A \cup S_u C &\subset S_u (A \cup C). \end{aligned}$$

In particular, the outradius of $S_u A$ is no larger than the outradius of A .

Consider a pair of non-empty cross sections $A(x)$ and $C(y)$. Let $a(x)$ be the leftmost point in $A(x)$, and let $c(y)$ be the rightmost point in $C(y)$. Clearly,

$$A(x) + C(y) \supset (a(x) + C(y)) \cup (A(x) + c(y)),$$

with equality when $A(x)$ and $C(y)$ are intervals. Since the two sets on the right hand side have only the point $a(x) + c(y)$ in common, the one-dimensional measure of $A(x) + C(y)$ is at least as large as the sum of the measures of $A(x)$ and $C(y)$. (This is the Brunn-Minkowski inequality in one dimension). It follows that $(A(x))^* + (C(y))^*$ is contained in $(A(x) + C(y))^*$, and therefore

$$(2) \quad S_u A + S_u C \subset S_u (A + C)$$

for every pair of compact sets $A, C \subset \mathbb{R}^n$. By definition of the Minkowski difference, this in turn implies that

$$(3) \quad S_u A - S_u C \supset S_u(A - C).$$

In particular, Steiner symmetrization reduces the volume of outer parallel sets and increases the volume of inner parallel sets.

In the proof of Theorem 1, we will bound the perimeter of $S_{u_n} \dots S_{u_1} A$ in terms of the volume of its parallel sets. Specifically, we will establish Eq. (1) for the **outer Minkowski perimeter**, given by

$$\text{Per}_M^+(A) = \limsup_{\delta \rightarrow 0} \frac{1}{\delta} (\text{Vol}(A + B_\delta) - \text{Vol}(A)),$$

and then argue that $\text{Per}_M^+(A) \geq \text{Per}(A)$ for every compact set A . The first lemma concerns the Minkowski sum and difference of a compact set A with a line segment.

Lemma 1. *Let $A \subset \mathbb{R}^n$ be a compact set, let u be a unit vector in \mathbb{R}^n , and fix $\beta > 0$. Assume that $S_u A = A$, and consider the line segment $L_{\beta u} = \{tu, |t| \leq \beta\}$. Then, for every $R > 0$,*

$$\begin{aligned} \text{Vol}((A + L_{\beta u}) \cap B_R) &\leq \text{Vol}(A \cap B_R) + 2\omega_{n-1}R^{n-1}\beta, \\ \text{Vol}((A - L_{\beta u}) \cap B_R) &\geq \text{Vol}(A \cap B_R) - 2\omega_{n-1}R^{n-1}\beta. \end{aligned}$$

Proof. By assumption, each non-empty cross section $A(x)$ is either a centered interval of positive length $\ell(x)$, or a single point, in which case we set $\ell(x) = 0$. The corresponding cross section of $A + L_{\beta u}$ is a line segment of length $\ell(x) + 2\beta$. The corresponding cross section of $A - L_{\beta u}$ is either a centered interval of length $\ell(x) - 2\beta$, a single point, or empty. The claims follow upon integration over $x \in u^\perp \cap B_R$. \square

Lemma 2. *Let $A \subset \mathbb{R}^n$ be a compact set, and let $R > 0$. For a finite collection of unit vectors $u_1, \dots, u_k \in \mathbb{R}^n$ and $\beta_1, \dots, \beta_k \geq 0$, set*

$$A_k = S_{u_k} \dots S_{u_1} A, \quad C_k = \sum_{i \leq k} S_{u_k} \dots S_{u_{i+1}} L_{\beta_i u_i},$$

where $L_{\beta_i u_i}$ is a line segment as in Lemma 1. Then

$$\begin{aligned} \text{Vol}((A_k + C_k) \cap B_R) &\leq \text{Vol}(A \cap B_R) + 2\omega_{n-1}R^{n-1} \sum_{i \leq k} \beta_i, \\ \text{Vol}((A_k - C_k) \cap B_R) &\geq \text{Vol}(A \cap B_R) - 2\omega_{n-1}R^{n-1} \sum_{i \leq k} \beta_i. \end{aligned}$$

Proof. We proceed by induction on k . Lemma 1 settles the base case $k = 1$. Let $1 < k \leq n$, and suppose the first claim holds for $k - 1$. By Eq. (2),

$$A_k + C_k \subset S_{u_k}(A_{k-1} + C_{k-1}) + L_{\beta_k u_k}.$$

We combine this with the first inequality of Lemma 1 and then apply the inductive hypothesis to obtain

$$\begin{aligned} \text{Vol}((A_k + C_k) \cap B_R) &\leq \text{Vol}(S_{u_k}((A_{k-1} + C_{k-1}) \cap B_R) + L_{\beta_k u_k}) \\ &\leq \text{Vol}((A_{k-1} + C_{k-1}) \cap B_R) + 2\omega_{n-1}R^{n-1}\beta_k \\ &\leq \text{Vol}(A \cap B_R) + 2\omega_{n-1}R^{n-1} \sum_{i \leq k} \beta_i. \end{aligned}$$

This completes the induction. For the second claim, we argue similarly, using Eq. (3) and the second inequality of Lemma 1. \square

The next lemma gives a lower bound for the inradius of the parallelepiped $C_n = \sum_{i \leq n} S_{u_n} \dots S_{u_{i+1}} L_{\beta u_i}$.

Lemma 3. *Let u_1, \dots, u_n be linearly independent unit vectors in \mathbb{R}^n , and let $\beta, \rho > 0$. If $\beta \det(u_1, \dots, u_k) \geq \rho$, then*

$$B_\rho \subset \sum_{i \leq n} S_{u_n} \dots S_{u_{i+1}} L_{\beta u_i}.$$

Proof. Denote by V_k the subspace spanned by u_1, \dots, u_k , and set

$$C_k = \sum_{i \leq k} S_{u_k} \dots S_{u_{i+1}} L_{\beta u_i}, \quad k = 1, \dots, n.$$

Let ρ_k be the inradius of C_k (considered as a subset of V_k), and let λ_k be the k -dimensional measure of the parallelepiped spanned by u_1, \dots, u_k . We will show by induction on k that $\rho_k \geq \beta \lambda_k$ for $k = 1, \dots, n$.

In the base case $k = 1$, we have $C_1 = L_{\beta u_1}$, $\rho_1 = \beta$, and $\lambda_1 = 1$. Let now $1 < k \leq n$, and suppose we have already shown that $\rho_{k-1} \geq \beta \lambda_{k-1}$. By definition,

$$C_k = S_{u_k} C_{k-1} + L_{\beta u_k}.$$

The Steiner symmetrization S_{u_k} acts on subsets of V_{k-1} as the orthogonal projection onto u_k^\perp . Let θ_k be the angle between V_{k-1} and u_k . The projection onto u_k^\perp shrinks the length of vectors in V_{k-1} by a factor that is no smaller than $\sin \theta_k$, and shrinks the $(k-1)$ -dimensional volume of subsets exactly by a factor $\sin \theta_k$. By the inductive assumption,

$$\rho_k \geq \rho_{k-1} \sin \theta_{k-1} \geq \beta \lambda_{k-1} \sin \theta_{k-1} = \beta \lambda_k,$$

completing the induction. \square

Theorem 2 (Volume estimate). *If $A \subset \mathbb{R}^n$ is a compact set with outradius R and u_1, \dots, u_n are linearly independent unit vectors in \mathbb{R}^n , then*

$$\begin{aligned} \text{Vol}(S_{u_n} \dots S_{u_1} A + B_\delta) &\leq \text{Vol}(A) + \frac{a_n(R + \delta)^{n-1}\delta}{|\det(u_1, \dots, u_n)|}, \\ \text{Vol}(S_{u_n} \dots S_{u_1} A - B_\delta) &\geq \text{Vol}(A) - \frac{a_n R^{n-1}\delta}{|\det(u_1, \dots, u_n)|} \end{aligned}$$

for every $\delta > 0$. Here, $a_n = 2n\omega_{n-1}$.

Proof. We apply Lemma 3 with $\beta = \delta/|\det(u_1, \dots, u_n)|$ to see that

$$B_\delta \subset \sum_{i \leq n} S_{u_n} \dots S_{u_{i+1}} L_{\beta u_i} =: C.$$

It follows from the first inequality in Lemma 2 that

$$\begin{aligned} \text{Vol}(S_{u_n} \dots S_{u_1} A + B_\delta) &\leq \text{Vol}((S_{u_n} \dots S_{u_1} A + C) \cap B_{R+\delta}) \\ &\leq \text{Vol}(A) + 2n\omega_{n-1}(R + \delta)^{n-1}\beta, \end{aligned}$$

proving the first claim. Similarly, we obtain from the second inequality in Lemma 2

$$\begin{aligned} \text{Vol}(S_{u_n} \dots S_{u_1} A - B_\delta) &\geq \text{Vol}((S_{u_n} \dots S_{u_1} A - C) \cap B_R) \\ &\geq \text{Vol}(A) - 2n\omega_{n-1}R^{n-1}\beta. \quad \square \end{aligned}$$

The next lemma is not needed for the proof of the main result. It will be used at the end of the paper to turn the volume estimate from Theorem 2 into an inequality for the volume radius of parallel sets.

Lemma 4. *Let A be a non-empty compact set in \mathbb{R}^n with $n \geq 2$. For $\delta > 0$, let $\rho(\delta)$ be the volume radius of $A + B_\delta$, let $\rho(-\delta)$ be the volume radius of $A - B_\delta$, and let r be the volume radius of A^* . Assume that*

$$\begin{aligned} \text{Vol}(A + B_\delta) &\leq \text{Vol}(A) + b(R + \delta)^{n-1}\delta, \\ \text{Vol}(A - B_\delta) &\geq \text{Vol}(A) - bR^{n-1}\delta \end{aligned}$$

for all $\delta > 0$, where $b \geq 2\omega_n r^n / R^n$ and $R \geq r$ are constants. Then

$$|\rho(\pm\delta) - r| \leq c\delta,$$

where $c = bR^{n-1}/(\omega_n r^{n-1})$.

Proof. Note that $c \geq 2r/R$. By Jensen's inequality,

$$\begin{aligned} \text{Vol}(A^* + B_{c\delta}) - \text{Vol}(A) &= n\omega_n \int_0^{c\delta} (r+s)^{n-1} ds \\ &\geq cn\omega_n r^{n-1} \delta \left(1 + \frac{c\delta}{2r}\right)^{n-1} \\ &\geq bR^{n-1} \delta \left(1 + \frac{\delta}{R}\right)^{n-1} \\ &\geq \text{Vol}(A + B_\delta) - \text{Vol}(A) \\ &= \text{Vol}(B_{\rho(\delta)}) - \text{Vol}(A), \end{aligned}$$

where the last two steps used the assumption on $A + B_\delta$ and the definition of $\rho(\delta)$. It follows that $A^* + B_{c\delta} \supset B_{\rho(\delta)}$, which gives the claim for $t > 0$. On the other hand, the assumption on $A - B_\delta$ implies that

$$\omega_n r^{n-1} (r - \rho(-\delta)) \leq \text{Vol}(B_{\rho(-\delta)}) \leq bR^{n-1} \delta,$$

which gives the $\rho(-\delta)$. \square

Proof of Theorem 1. The first inequality of Theorem 2 yields for the outer Minkowski perimeter

$$\begin{aligned} \text{Per}_M^+(S_{u_n} \dots S_{u_1} A) &= \limsup_{\delta \rightarrow 0} \frac{1}{\delta} (\text{Vol}(S_{u_n} \dots S_{u_1} A + B_\delta) - \text{Vol}(A)) \\ &\leq \frac{a_n R^{n-1}}{|\det(u_1, \dots, u_n)|}. \end{aligned}$$

The proof is completed with the lemma below. \square

Lemma 5. *If $A \subset \mathbb{R}^n$ is compact, then $\text{Per}(A) \leq \text{Per}_M^+(A)$.*

Proof (L. Ambrosio). Apply the coarea formula (see [4, Theorem 13.1]) to the function $f(x) = \text{dist}(x, A)$, which is clearly Lipschitz continuous, and hence differentiable almost everywhere. Since $|\nabla f| = 1$ a.e. outside A and vanishes a.e. on A ,

$$\begin{aligned} \text{Vol}(A + B_\delta) - \text{Vol}(A) &= \int_{A+B_\delta} |\nabla f(x)| dx \\ &= \int_0^\delta \text{Per}(A + B_t) dt \\ &\geq \delta \cdot \inf_{0 < t < \delta} \text{Per}(A + B_t). \end{aligned}$$

In the second step, we have used the coarea formula and observed that $f^{-1}(t) = \partial(A + B_t)$ for $t > 0$. We now divide by δ and take $\delta \rightarrow 0$. Since A is compact, the parallel set $A + B_\delta$ converges to A in

symmetric difference. It follows from the lower semicontinuity of the perimeter that

$$\begin{aligned} \text{Per}_M^+(A) &= \limsup_{\delta \rightarrow 0} \frac{1}{\delta} (\text{Vol}(A + B_\delta) - \text{Vol}(A)) \\ &\geq \liminf_{\delta \rightarrow 0} \text{Per}(A + B_\delta) \\ &\geq \text{Per}(A). \end{aligned}$$

This concludes the proof of the main result. \square

There are various notions of perimeter, which agree for open sets with smooth boundary but may differ for less regular sets (see [1] for some recent results). In particular, $\text{Per}(A)$ can be much smaller than the $(n-1)$ -dimensional Hausdorff measure of the topological boundary of A . Having established Eq. (1) for the Caccioppoli perimeter, we wish to extend the inequality to another commonly used measure of the size of the boundary.

The **two-sided Minkowski perimeter** of a compact set A is defined by

$$\text{Per}_M(A) = \limsup_{\delta \rightarrow 0} \frac{1}{2\delta} \text{Vol}(\partial A + B_\delta),$$

where ∂A is the boundary of A . It is not hard to show, using a Vitali covering argument, that the $(n-1)$ -dimensional Hausdorff measure of ∂A is bounded by $2 \cdot 3^{n-1}(\omega_{n-1}/\omega_n) \text{Per}_M(A)$, but it is not clear (to us) whether the bound holds without the constant factor. The last lemma will be used to relate ∂A to the outer and inner parallel sets of A .

Lemma 6. *For any pair of compact sets $A, C \subset \mathbb{R}^n$ and every $\delta > 0$,*

$$\partial A + C \subset (A + C) \setminus \text{interior}(A - C^-),$$

where $C^- = \{-c : c \in C\}$ is the reflection of C through the origin. If C is connected, then the converse inclusion also holds.

Proof. Clearly, $\partial A + C \subset A + C$. We need to show that $\partial A + C$ does not intersect the interior of $A - C^-$. Suppose that x lies in the interior of $A - C^-$. Then there exists $\delta > 0$ such that $B_\delta(x) \subset A - C^-$. This means that $B_\delta(x - c) \subset A$, i.e., $\text{dist}(x - c, \partial A) \geq \delta$ for every $c \in C$. We conclude that x cannot lie in $\partial A + C$.

For the reverse inclusion, assume furthermore that C is connected. Let $x \in (A + C) \setminus \text{interior}(A - C^-)$, and consider

$$\begin{aligned} C_1 &= \{c \in C : x - c \in A\}, \\ C_2 &= \{c \in C : x - c \notin \text{interior}(A)\}. \end{aligned}$$

By definition, C_1 and C_2 are closed and cover C . Furthermore, C_1 is non-empty because $x \in A + C$, and C_2 is non-empty because $x \notin \text{interior}(A - C^-)$. Since C is connected, C_1 and C_2 cannot be disjoint. Pick $c \in C_1 \cap C_2$. Then $x - c \in \partial A$, i.e., $x \in \partial A + C^-$, as claimed. \square

In the special case where $C = B_\delta$, the lemma implies that

$$(4) \quad \text{Vol}(\partial A + B_\delta) = \text{Vol}(A + B_\delta) - \text{Vol}(A - B_\delta),$$

because the boundary of $A - B_\delta$, which consists of all points having distance exactly δ from the complement of A , is a set of volume zero. Combining Eq. (4) with Theorem 2, we obtain

$$\begin{aligned} \text{Vol}(\partial S_{u_n} \dots S_{u_1} A + B_\delta) &= \text{Vol}(S_{u_n} \dots S_{u_1} A + B_\delta) - \text{Vol}(S_{u_n} \dots S_{u_1} A - B_\delta) \\ &\leq \frac{4n\omega_{n-1}(R + \delta)^{n-1}\delta}{|\det(u_1, \dots, u_n)|}. \end{aligned}$$

Dividing by 2δ and taking $\delta \rightarrow 0$ extends Eq. (1) to the two-sided Minkowski perimeter.

Corollary 1. *Under the assumptions of Theorem 1,*

$$\text{Per}_M(S_{u_n} \dots S_{u_1} A) \leq \frac{a_n R^{n-1}}{|\det(u_1, \dots, u_n)|}.$$

Since $\text{Per}(A) \leq \text{Per}_M(A)$ by the same reasoning as in Lemma 5, this improves upon Theorem 1.

Finally, we discuss an application to random sequences of Steiner symmetrizations. Consider a non-empty compact set $A \subset \mathbb{R}^n$, let r be its volume radius, and assume that $A \subset B_R$. Let $\{U_k\}_{k \geq 0}$ be a sequence of unit vectors picked independently, uniformly at random from the unit sphere in \mathbb{R}^n , and define recursively

$$A_0 = A, \quad A_{k+1} = S_{U_k} A_k \quad (k \geq 0).$$

It was recently shown by Burchard and Fortier that the expectation of the symmetric difference from A_k to A^* satisfies

$$(5) \quad E(A_k \triangle A^*) \leq n\omega_n 2^{n+1} R^n k^{-1}$$

for all $k > 0$ [2, Proposition 5.2]. Under suitable regularity assumptions on ∂A , this can be used to bound the Hausdorff distance $d_H(\partial A_k, \partial A^*)$, which controls how much the outradius and inradius of A_k differ from its volume radius.

We briefly describe the tools developed in [2, Section 7]. The authors consider the auxiliary function

$$f(x) = \text{dist}(x, \mathbb{R}^n \setminus A) + (R - \text{dist}(x, A))_+$$

and its symmetrizations

$$F_0 = f, \quad F_{k+1} = S_{U_k} F_k \quad (k \geq 0).$$

By construction, $A_k = \{x : F_k(x) > R\}$. Using Eqs. (2) and (3), they show that

$$(6) \quad d_H(\partial A_k, \partial A^*) \leq \max_{\pm} |\rho(\pm \|F_k - f^*\|_{\infty}) - r|,$$

where $\rho(\pm\delta)$ is the volume radius of the parallel set $A \pm B_{\delta}$. It follows from [2, Proposition 5.2] that

$$(7) \quad E(\|F_k - f^*\|_{\infty}) \leq 12Rk^{-\frac{1}{n+1}}$$

for $k > 0$. Under the assumption that A has finite Minkowski perimeter, they differentiate ρ at $\delta = 0$ and obtain from Eqs. (6) and (7) a sequence of Steiner symmetrizations along which $d_H(\partial A_k, \partial A^*) = O(k^{-\frac{1}{n+1}})$ as $k \rightarrow \infty$. The rate of convergence estimates in Eqs. (5) and (7) are proved by comparing Steiner symmetrization with polarization, a simpler rearrangement that preserves perimeter as well as volume [2, Section 5].

We will use Theorems 1 and 2 to obtain a stronger bound on the perimeter of A_{n+1} that results in stronger bounds on ρ and, through Eq. (6), on $d_H(\partial A_k, \partial A^*)$. By Theorem 1, the perimeter of A_n is almost surely finite, because the probability that the vectors U_0, \dots, U_{n-1} lie in a common hyperplane is zero. We next argue that $\text{Per}(A_{n+1})$ has finite expectation. Since $\text{Per}(A_{n+1}) \leq \text{Per}(A_n)$, we can apply Theorem 1 to A_n and A_{n+1} to obtain

$$\text{Per}(A_{n+1}) \leq a_n R^{n-1} Y_n,$$

where $a_n = 2n\omega_{n-1}$, and the random variable Y_n is given by

$$(8) \quad Y_n = \min \{ |\det(U_1, \dots, U_n)|^{-1}, |\det(U_0, \dots, U_{n-1})|^{-1} \}.$$

As in the proof of Lemma 3, we expand $|\det(U_1, \dots, U_n)| = \prod_{k=2}^n X_k$ and $|\det(U_0, \dots, U_{n-1})| = X'_n \cdot \prod_{k=2}^{n-1} X_k$, where X_k is the Euclidean distance of U_k to the subspace of \mathbb{R}^n spanned by U_1, \dots, U_{k-1} , and X'_n is the distance of U_0 to the subspace spanned by U_1, \dots, U_{n-1} . Then

$$Y_n = (\max\{X_n, X'_n\})^{-1} \cdot \prod_{k=2}^{n-1} X_k^{-1}.$$

By rotational invariance, X_k has the same distribution as the distance of a random point on the sphere from a $(k-1)$ -dimensional coordinate plane, X'_n has the same distribution as X_n , and X_2, \dots, X_n, X'_n are independent. Since the sphere is compact and intersects the coordinate planes transversally, there exist constants $b_{n,k}$ such that $P(X_k \leq t) \leq$

$b_{n,k}t^{n-k+1}$ for $2 \leq k \leq n$. By the independence of X_n and X'_n , it follows that $P(\max\{X_n, X'_n\} \leq t) \leq (b_{n,n}t)^2$. Therefore,

$$(9) \quad E(Y_n) = E((\max\{X_n, X'_n\})^{-1}) \cdot \prod_{k=2}^{n-1} E(X_k^{-1}) < \infty.$$

We have proved the following inequality.

Corollary 2.

$$E(\text{Per}(A_{n+1})) \leq b_n R^{n-1},$$

where $b_n = 2n\omega_{n-1}E(Y_n)$ depends only on the dimension.

We want to apply Eqs. (6) and (7) to the conditional expectation $E(\cdot \mid A_{n+1})$. Let \tilde{f} , \tilde{F}_k , and $\tilde{\rho}$ be the functions corresponding to f , F_k , and ρ with A_{n+1} in place of A . Replacing Theorem 1 with Theorem 2 in the proof of Corollary 2, we obtain for every $\delta > 0$,

$$\text{Vol}(A_{n+1} + B_\delta) \leq \text{Vol}(A) + a_n(R + \delta)^{n-1}\delta Y_n,$$

$$\text{Vol}(A_{n+1} + B_\delta) \geq \text{Vol}(A) - a_n R^{n-1}\delta Y_n,$$

where $a_n = 2n\omega_{n-1}$, and Y_n is the random variable from Eq. (8). Since $a_n \geq 2\omega_n$, $Y_n \geq 1$, and $R \geq r$, the assumptions of Lemma 4 are satisfied with $b = a_n Y_n$, and so

$$|\tilde{\rho}(\pm\delta) - r| \leq \frac{a_n R^{n-1}}{\omega_n r^{n-1}} \delta.$$

It follows that

$$\begin{aligned} d_H(\partial A_{n+1+k}, \partial A^*) &= \max_{\pm} |\tilde{\rho}(\pm \|\tilde{F}_k - f^*\|_\infty) - r| \\ &\leq \frac{a_n R^{n-1}}{\omega_n r^{n-1}} Y_n \|\tilde{F}_k - f^*\|_\infty \end{aligned}$$

for $k > 0$, see Eq. (6). Since Y_n is independent of U_k for $k > n$ and \tilde{F}_k depends on U_0, \dots, U_n only through A_{n+1} , we can invoke the Markov property to obtain

$$\begin{aligned} E(d_H(\partial A_{n+1+k}, \partial A^*)) &= E(E(d_H(\partial A_{n+1+k}, \partial A^*) \mid U_0, \dots, U_n)) \\ &\leq \frac{a_n R^{n-1}}{\omega_n r^{n-1}} E(Y_n E(\|\tilde{F}_k - f^*\|_\infty \mid A_{n+1})) \\ &\leq \frac{12a_n R^{n-1}}{\omega_n r^{n-1}} E(Y_n) R k^{-\frac{1}{n+1}} \end{aligned}$$

for $k > 0$. In the last line, we have applied Eq. (7) to \tilde{F}_k . By Corollary 2, the expected value of Y_n is finite. We shift the index and adjust the constant to obtain the desired bound on the rate of convergence.

Corollary 3.

$$E(d_H(\partial A_k, \partial A^*)) \leq c_n (R/r)^{n-1} R k^{-\frac{1}{n+1}},$$

where $c_n = 25n\omega_{n-1}E(Y_n)/\omega_n$.

We close with an explicit bound on the constants b_n and c_n that appear in Corollaries 2 and 3. We consider separately each of the expected values in Eq. (9). A routine spherical integral (conveniently evaluated as a Gaussian integral over \mathbb{R}^n) gives $E(X_k^{-1}) = \frac{(n-k+1)\omega_{n-k+1}}{(n-k)\omega_{n-k}} \cdot \frac{(n-1)\omega_{n-1}}{n\omega_n}$ for $2 \leq k < n$. Collecting terms, we obtain

$$\prod_{k=2}^{n-1} E(X_k^{-1}) = \frac{((n-1)\omega_{n-1})^{n-1}}{2(n\omega_n)^{n-2}}.$$

A similar integral yields $P(X_n \leq \sin \alpha) = \frac{(n-1)\omega_{n-1}}{n\omega_n} \int_{-\alpha}^{\alpha} (\cos t)^{n-2} dt$. Using that X_n and X'_n are independent, we estimate for $n \geq 3$

$$E((\max\{X_n, X'_n\})^{-1}) \leq 1 + \left(\frac{2(n-1)\omega_{n-1}}{n\omega_n} \right)^2.$$

When $n = 3$, this equation holds with equality, resulting in $E(Y_3) = \pi$. In two dimensions, we find that $E(Y_2) \leq 2$, and for $n \rightarrow \infty$, we have $\lim n^{-1} \log E(Y_n) = \sqrt{2e}$. Since $\lim n^{-1} \log(n\omega_{n-1}) = -\infty$ and $\lim n^{-1} \log(n\omega_{n-1}/\omega_n) = 1$, we conclude that b_n converges to zero and c_n grows exponentially.

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